

Asymptotic Behavior of Global Classical Solutions of Quasilinear Hyperbolic Systems with Weakly Linear Degeneracy

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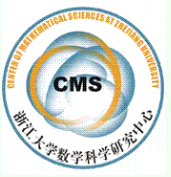
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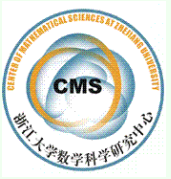
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Abstract

In this paper, we study the asymptotic behavior of global classical solutions of the Cauchy problem for general quasilinear hyperbolic systems with weakly linearly degenerate characteristic fields. Based on the existence results on the global classical solution proved by Zhou, we prove that, when t tends to infinity, the solution approaches a combination of C^1 travelling wave solutions, provided that the total variation and L^1 norm of the initial data are sufficiently small.

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1 Introduction and main result

Consider the following first order quasilinear strictly hyperbolic system

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} &= 0, \\ t = 0 : \quad u(0, x) &= f(x), \end{aligned} \quad (1.1)$$

where $A(u) = (a_{ij}(u)) \in C^3$.

Strict hyperbolicity: $\lambda_1(u) < \cdots < \lambda_n(u)$.

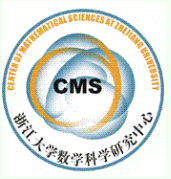
By F. John (CPAM, 1974), all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ ($i, j = 1, \cdots, n$) are C^3 smooth and satisfy the normalized condition

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \quad (1.2)$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n). \quad (1.3)$$

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Definition (D.Q.Li, Y.Zhou and D.X.Kong, CPDE, 1994)
The i -th characteristic $\lambda_i(u)$ is **weakly linearly degenerate**, if, along the i -th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\frac{du}{ds} = r_i(u), \quad u(0) = 0, \quad (1.4)$$

we have

$$\nabla \lambda_i(u) r_i(u) \equiv 0, \quad \forall |u| \text{ small}, \quad (1.5)$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall |s| \text{ small}. \quad (1.6)$$

If all characteristics are weakly linearly degenerate, then system (1.1) is called to be weakly linearly degenerate.

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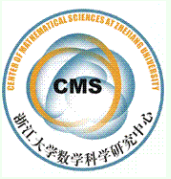
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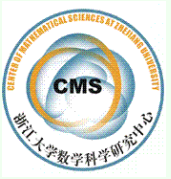
By D.Q.Li, Y.Zhou and D.X.Kong (CPDE, 1994), there exists an invertible C^4 transformation $u = u(\tilde{u})$ ($u(0) = 0$) such that in the $\tilde{u}$ -space,

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \quad \forall |\tilde{u}_i| \text{ small} \quad (i = 1, \dots, n). \quad (1.7)$$

Such a transformation is called the **normalized transformation** and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)^T$ are called the **normalized variables** or **normalized coordinates**. In the normalized coordinates, (1.6) simply reduces to

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall |u_i| \text{ small}. \quad (1.6a)$$

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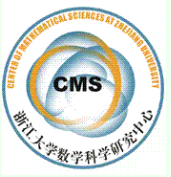


If the initial data $f(x)$ satisfies the following decay property:

$$\varrho \triangleq \sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu} (|f(x)| + |f'(x)|)\} < +\infty \quad (\mu > 0) \quad (1.8)$$

is sufficiently small, D.Q.Li, Y.Zhou and D.X.Kong proved that the Cauchy problem (1.1) admits a unique global classical solution, provided that the system (1.1) is weakly linearly degenerate. The condition $\mu > 0$ is essential. If $\mu = 0$, a counterexample constructed in Kong (Chin. Ann. Math. 21B, 2000) showing that the classical solution may blow up in a finite time, even when the system (1.1) is weakly linearly degenerate.

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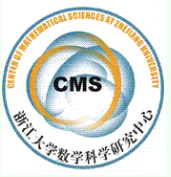
Theorem A.(Zhou Yi, Chin. Ann. Math., 25B, 2004) Suppose that the system (1.1) is strictly hyperbolic and weakly linearly degenerate. Suppose furthermore that $A(u) \in C^2$ and $f \in C^1$ with bounded C^1 norm. Let

$$M = \sup_{x \in \mathbb{R}} |f'(x)| < +\infty. \quad (1.9)$$

Then there exists a small constant $\epsilon > 0$ such that the Cauchy problem (1.1) admits a unique global C^1 solution $u = u(t, x)$ for all $t \in \mathbb{R}$, provided that

$$\int_{-\infty}^{+\infty} |f'(x)| dx \leq \epsilon, \quad \int_{-\infty}^{+\infty} |f(x)| dx \leq \frac{\epsilon}{M}. \quad (1.10)$$

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Based on Theorem A, we prove the following theorem.

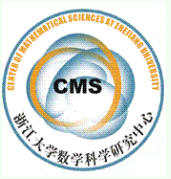
Theorem 1.1 (Asymptotic Behavior). Under the assumptions of Theorem A and $A(u) \in C^3$, there exists a unique C^1 vector-valued function $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))^T$ such that

$$u(t, x) \longrightarrow \sum_{i=1}^n \varphi_i(x - \lambda_i(0)t) e_i, \quad \text{as } t \rightarrow +\infty; \quad (1.11)$$

moreover, there exists a positive constant κ_1 independent of ϵ , M , x_1 and x_2 such that for every $i \in \{1, \dots, n\}$,

$$|\varphi_i(x_1) - \varphi_i(x_2)| \leq \kappa_1 M |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}. \quad (1.12)$$

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Furthermore, if $f'(x)$, the derivative of the initial data, is globally Lipschitz continuous, that is, there exists a positive constant ς such that

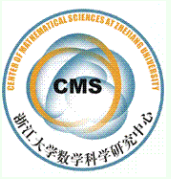
$$|f'(x_1) - f'(x_2)| \leq \varsigma |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}, \quad (1.13)$$

then

$$|\varphi'(x_1) - \varphi'(x_2)| \leq \kappa_2(\varsigma + M^2)|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}, \quad (1.14)$$

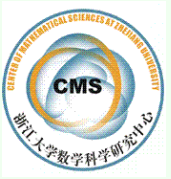
where κ_2 is a positive constant independent of ϵ , M , ς , x_1 and x_2 .

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Remark Theorem 1.1 gives the exact time asymptotic behavior of the global classical solutions presented in Theorem A. For the initial data satisfying the decay property (1.8), Kong and Yang (CPDE, 28, 2003) proved that, when t tends to infinity, the global classical solution approaches a combination of C^1 travelling wave solutions at algebraic rate $(1+t)^{-\mu}$. Comparing with it, because of the lack of the decay rate of the initial data, in the present situation there is no any estimate on the convergence rate.

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The paper is organized as follows. For the sake of completeness, in Section 2 we recall John's formula on the decomposition of waves with some supplements. Section 3 is devoted to establishing some new uniform estimates. The main result, Theorem B, is proved in Section 4. By analyzing carefully the global propagation properties of the classical waves, we use the estimates given in Section 3 to describe the large time behavior of the global classical solutions, and then construct the desired travelling waves.

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2 Preliminaries

Resolution of wave (F.John, CPAM, 1974):

Let

$$v_i(u) = l_i(u)u, \quad w_i(u) = l_i(u)u_x \quad (i = 1, \dots, n), \quad (2.1)$$

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u) \frac{\partial}{\partial x}. \quad (2.2)$$

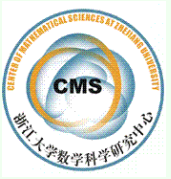
We have

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^n \beta_{ijk}(u) v_j w_k \triangleq F_i(t, x), \quad (2.3)$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u) v_i)}{\partial x} = \sum_{j,k=1}^n \tilde{\beta}_{ijk}(u) v_j w_k \triangleq \tilde{F}_i(t, x), \quad (2.4)$$

$$d[v_i(dx - \lambda_i(u)dt)] = \tilde{F}_i(t, x)dt dx. \quad (2.5)$$

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$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k \triangleq G_i(t, x), \quad (2.6)$$

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^n \tilde{\gamma}_{ijk}(u) w_j w_k \triangleq \tilde{G}_i(t, x), \quad (2.7)$$

$$d[w_i(dx - \lambda_i(u)dt)] = \tilde{G}_i(t, x)dt dx. \quad (2.8)$$

When the system is weakly linearly degenerate, in the normalized coordinates, it holds that

$$\tilde{\beta}_{ijj}(u_j e_j) \equiv 0, \quad \gamma_{iii}(u_i e_i) \equiv 0, \quad \forall i \in \{1, \dots, n\}. \quad (2.9)$$

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3 Uniform estimates

In this section, we shall establish some new uniform estimates which play a key role in the proof of Theorem 1.1.

For any fixed $T \geq 0$, we introduce

$$U_{\infty}(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |u(t, x)|, \quad V_{\infty}(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |v(t, x)|,$$

$$W_{\infty}(T) = \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |w(t, x)|,$$

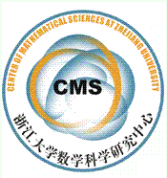
$$V_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |v(t, x)| dx,$$

$$W_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |w(t, x)| dx,$$

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$$\tilde{U}_1(T) = \max_{j \neq i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |u_i(t, x)| dt, \quad \tilde{V}_1(T) = \max_{j \neq i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |v_i| dt,$$

$$\tilde{W}_1(T) = \max_{j \neq i} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |w_i(t, x)| dt,$$

$$\bar{U}_1(T) = \max_{j \neq i} \sup_{L_j} \int_{L_j} |u_i(t, x)| dt, \quad \bar{V}_1(T) = \max_{j \neq i} \sup_{L_j} \int_{L_j} |v_i| dt,$$

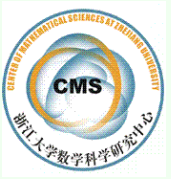
$$\bar{W}_1(T) = \max_{j \neq i} \sup_{L_j} \int_{L_j} |w_i(t, x)| dt,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$, $\tilde{C}_j$ stands for any given j -th characteristic on the domain $[0, T] \times \mathbb{R}$, while L_j stands for any given ray with the slope $\lambda_j(0)$.

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Combining Lemma 4.1 and (4.52) in Zhou Yi we get

Lemma 3.1 Under the assumptions of Theorem 1.1, there exists a positive constant K_1 independent of ϵ , M and T such that

$$V_1(T), \tilde{V}_1(T) \leq K_1 \frac{\epsilon}{M},$$

$$W_1(T), \tilde{W}_1(T) \leq K_1 \epsilon,$$

$$U_\infty(T), V_\infty(T) \leq K_1 \epsilon,$$

$$W_\infty(T) \leq K_1 M.$$

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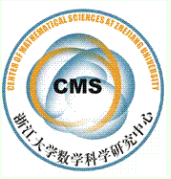
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On the other hand, we have

Lemma 3.2 Under the assumptions of Theorem 1.1, there exists a positive constant K_2 independent of ϵ , M and T such that

$$\tilde{U}_1(T) \leq K_2 \frac{\epsilon}{M}, \quad (3.1)$$

$$\bar{U}_1(T), \bar{V}_1(T) \leq K_2 \frac{\epsilon}{M} \quad (3.2)$$

and

$$\bar{W}_1(T) \leq K_2 \epsilon. \quad (3.3)$$

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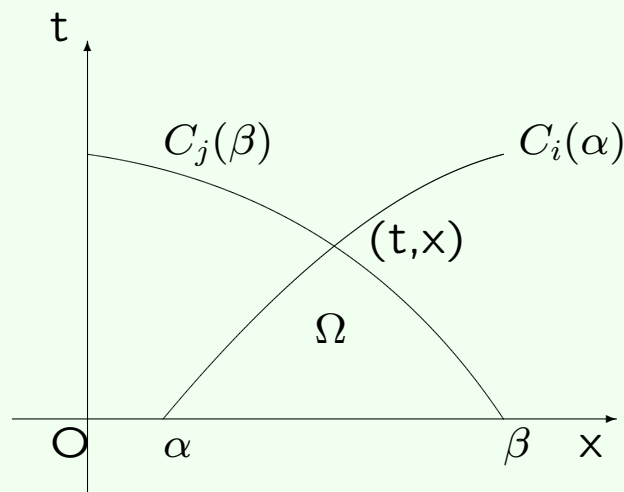


Figure 1

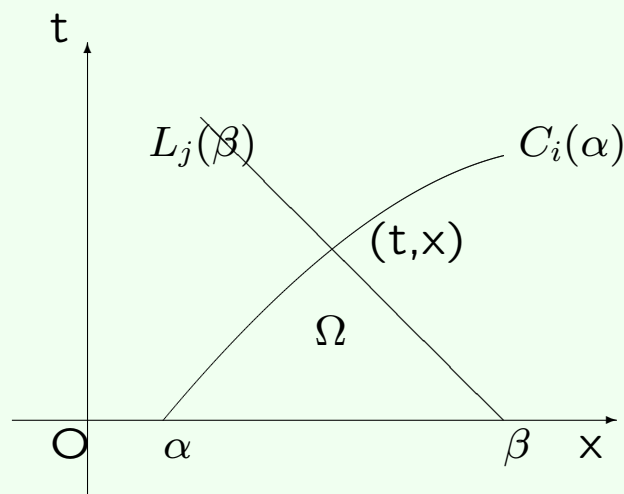


Figure 2



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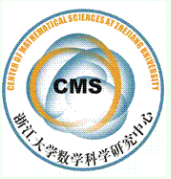
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Combining Lemmas 3.1-3.2 gives

Lemma 3.3 Under the assumptions of Theorem 1.1, there exists a positive constant K_3 independent of ϵ and M such that

$$V_1(\infty), \bar{U}_1(\infty), \tilde{U}_1(\infty), \bar{V}_1(\infty), \tilde{V}_1(\infty) \leq K_3 \frac{\epsilon}{M},$$

$$W_1(\infty), \bar{W}_1(\infty), \tilde{W}_1(\infty) \leq K_3 \epsilon,$$

$$U_\infty(\infty), V_\infty(\infty) \leq K_3 \epsilon, \quad W_\infty(\infty) \leq K_3 M,$$

where

$$V_1(\infty) = \sup_{0 \leq t \leq \infty} \int_{-\infty}^{+\infty} |v(t, x)| dx,$$

etc.

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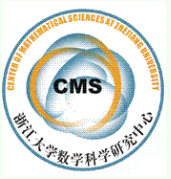
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Lemma 3.4 Under the assumptions of Theorem 1.1, for any $t \in \mathbb{R}^+$ and arbitrary $\alpha, \beta \in \mathbb{R}$, it holds that

$$|u(t, \alpha + \lambda_i(0)t) - u(t, \beta + \lambda_i(0)t)| \leq c_1 M |\alpha - \beta|;$$

moreover, for any given C^1 function $g(u)$,

$$|g(u(t, \alpha + \lambda_i(0)t)) - g(u(t, \beta + \lambda_i(0)t))| \leq c_2 M |\alpha - \beta|$$

and

$$|g(u(t, x_i(t, \alpha))) - g(u(t, x_i(t, \beta)))| \leq c_3 M |\alpha - \beta|,$$

where c_i ($i = 1, 2, \dots$) stand for some positive constants independent of ϵ , M , t , α and β .

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For any fixed $T \geq 0$ and for arbitrary $\alpha, \beta \in \mathbb{R}$, we introduce

$$U_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |u_j(s, \alpha + \lambda_i(0)s) - u_j(s, \beta + \lambda_i(0)s)| ds,$$

$$V_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |v_j(s, \alpha + \lambda_i(0)s) - v_j(s, \beta + \lambda_i(0)s)| ds,$$

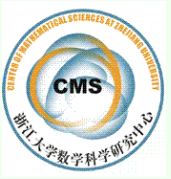
$$W_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |w_j(s, \alpha + \lambda_i(0)s) - w_j(s, \beta + \lambda_i(0)s)| ds,$$

$$\tilde{U}_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |u_j(s, x_i(s, \alpha)) - u_j(s, x_i(s, \beta))| ds,$$

$$\tilde{V}_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |v_j(s, x_i(s, \alpha)) - v_j(s, x_i(s, \beta))| ds,$$

$$\tilde{W}_{\alpha}^{\beta}(T) = \max_{j \neq i} \int_0^T |w_j(s, x_i(s, \alpha)) - w_j(s, x_i(s, \beta))| ds.$$

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Lemma 3.5 Under the assumptions of Theorem 1.1, there exists a positive constant K_4 independent of ϵ , M , T , α and β such that

$$U_{\alpha}^{\beta}(T) \leq K_4 \epsilon |\alpha - \beta|, \quad (3.4)$$

$$V_{\alpha}^{\beta}(T) \leq K_4 \epsilon |\alpha - \beta| \quad (3.5)$$

and

$$W_{\alpha}^{\beta}(T) \leq K_4 M |\alpha - \beta|. \quad (3.6)$$

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Proof.

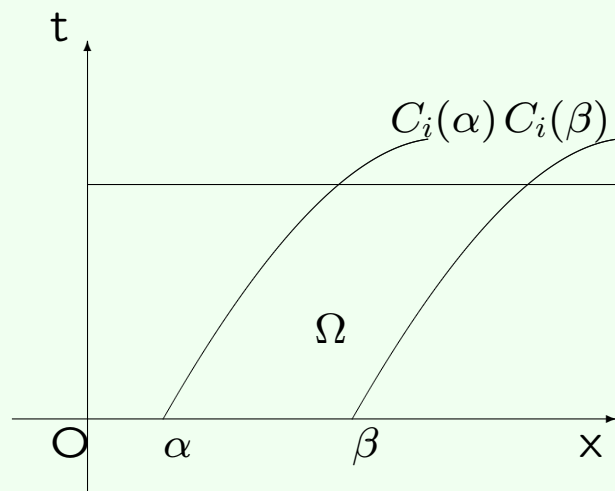


Figure 3

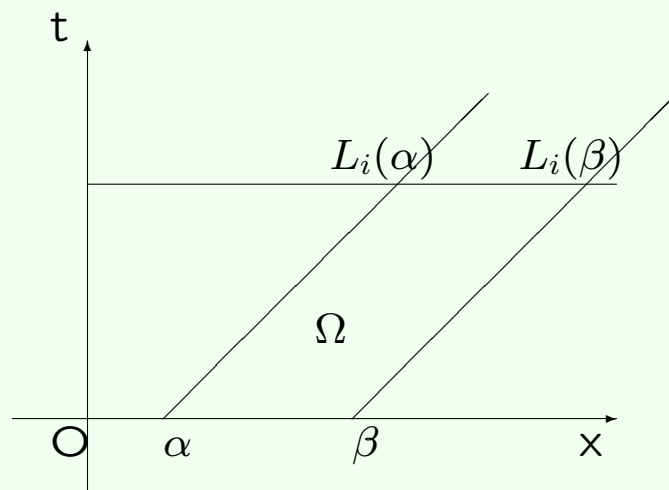


Figure 4



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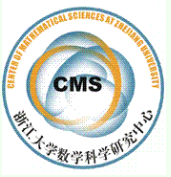
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It follows from (2.5) that

$$d[\xi(t, x)v_j(dx - \lambda_j(u)dt)] = \xi(t, x)\tilde{F}_j(t, x)dtdx \quad \text{a.e.},$$

where

$$\xi(t, x) = \text{sign}\{[v_j(t, \beta + \lambda_i(0)t) - v_j(t, \alpha + \lambda_i(0)t)] \bullet [(\lambda_i(0) - \lambda_j(u(t, \beta + \lambda_i(0)t)))]\}.$$

By Green formula, we have

$$\begin{aligned} \int \int_{\tilde{\Omega}} \xi(t, x)\tilde{F}_j(s, x)dsdx &= \int_0^T [\xi v_j(\lambda_i(0) - \lambda_j(u))](s, \beta + \lambda_i(0)s)ds - \\ &\quad \int_0^T [\xi v_j(\lambda_i(0) - \lambda_j(u))](s, \alpha + \lambda_i(0)s)ds + \\ &\quad \int_{\alpha}^{\beta} (\xi v_j)(0, x)dx - \int_{\alpha}^{\beta} (\xi v_j)(T, x + \lambda_i(0)T)dx. \end{aligned} \quad (3.7)$$

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When $i \neq j$, we obtain from (3.7) that

$$\begin{aligned}
 & \int_0^T |v_j(s, \alpha + \lambda_i(0)s) - v_j(s, \beta + \lambda_i(0)s)| ds \\
 & \leq \frac{1}{\delta_0} \{ [2V_\infty(T) + c_2 M \bar{V}_1(T)] |\alpha - \beta| + \int \int_{\tilde{\Omega}} |\tilde{F}_j(s, x)| ds dx \} \\
 & \leq c_4 \{ \epsilon |\alpha - \beta| + \int \int_{\tilde{\Omega}} |\tilde{F}_j(s, x)| ds dx \}.
 \end{aligned} \tag{3.8}$$

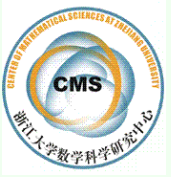
Using Lemma 3.1-3.2, we have

$$\begin{aligned}
 \int \int_{\tilde{\Omega}} |\tilde{F}_j(s, x)| ds dx & \leq c_5 \sum_{k=1}^n \sum_{l \neq k} \int \int_{\tilde{\Omega}} |v_k w_l| ds dx \\
 & = c_5 \sum_{k=1}^n \sum_{l \neq k} \int_\alpha^\beta dx \int_0^T |v_k w_l|(s, x + \lambda_i(0)s) ds \\
 & \leq c_6 \epsilon |\alpha - \beta|.
 \end{aligned} \tag{3.9}$$

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Substituting (3.9) into (3.8) gives

$$\int_0^T |v_j(s, \alpha + \lambda_i(0)s) - v_j(s, \beta + \lambda_i(0)s)| ds \leq c_7 \epsilon |\alpha - \beta|. \quad (3.10)$$

This proves (3.5). We can prove (3.6) similarly by (2.8).

Similarly, we can prove the following lemma.

Lemma 3.6 Under the assumptions of Theorem 1.1, there exists a positive constant K_5 independent of ϵ , M , T , α and β such that

$$\tilde{U}_\alpha^\beta(T) \leq K_5 \epsilon |\alpha - \beta|,$$

$$\tilde{V}_\alpha^\beta(T) \leq K_5 \epsilon |\alpha - \beta|$$

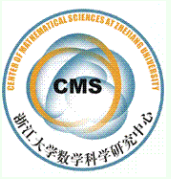
and

$$\tilde{W}_\alpha^\beta(T) \leq K_5 M |\alpha - \beta|.$$

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Combining Lemmas 3.5-3.6 gives

Lemma 3.7 Under the assumptions of Theorem 1.1, there exists a positive constant K_6 independent of ϵ , M , α and β such that

$$U_{\alpha}^{\beta}(\infty), \tilde{U}_{\alpha}^{\beta}(\infty) \leq K_6 \epsilon |\alpha - \beta|,$$

$$V_{\alpha}^{\beta}(\infty), \tilde{V}_{\alpha}^{\beta}(\infty) \leq K_6 \epsilon |\alpha - \beta|$$

and

$$W_{\alpha}^{\beta}(\infty), \tilde{W}_{\alpha}^{\beta}(\infty) \leq K_6 M |\alpha - \beta|,$$

where

$$U_{\alpha}^{\beta}(\infty) = \max_{j \neq i} \int_0^{\infty} |u_j(s, \alpha + \lambda_i(0)s) - u_j(s, \beta + \lambda_i(0)s)| ds,$$

etc.

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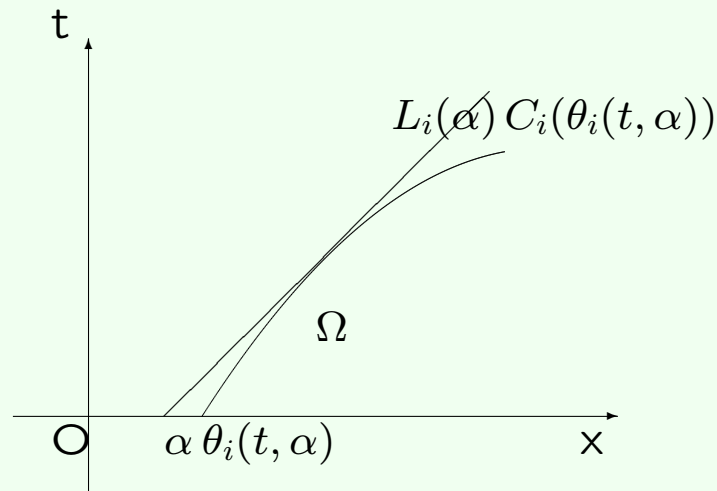


Figure 5

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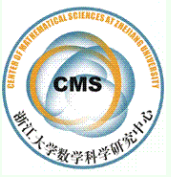
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For any fixed $(t, \alpha + \lambda_i(0)t)$, there exists a unique $\theta_i(t, \alpha) \in \mathbb{R}$ such that

$$\theta_i(t, \alpha) + \int_0^t \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))) ds = \alpha + \lambda_i(0)t,$$

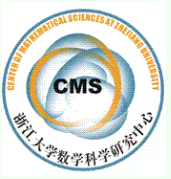
namely,

$$\theta_i(t, \alpha) = \alpha + \int_0^t [\lambda_i(0) - \lambda_i(u(s, x_i(s, \theta_i(t, \alpha))))] ds,$$

where $x = x_i(s, \theta_i(t, \alpha))$ stands for the i -th characteristic passing through the point $(0, \theta_i(t, \alpha))$, which is defined by

$$\frac{dx_i(s, \theta_i(t, \alpha))}{ds} = \lambda_i(u(s, x_i(s, \theta_i(t, \alpha)))) , \quad x_i(0, \theta_i(t, \alpha)) = \theta_i(t, \alpha).$$

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Lemma 3.8 Under the assumptions of Theorem 1.1, for any given $\alpha \in \mathbb{R}$ there exists a unique $\vartheta_i(\alpha)$ such that $\theta_i(t, \alpha)$ converges to $\vartheta_i(\alpha)$ when t tends to ∞ ; moreover, $\vartheta_i(\alpha)$ satisfies

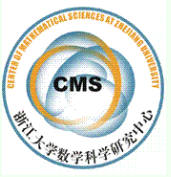
$$|\vartheta_i(\alpha) - \alpha| \leq K_7 \frac{\epsilon}{M},$$

and is a globally Lip-continuous function of α , more precisely, the following estimate holds

$$|\vartheta_i(\alpha) - \vartheta_i(\beta)| \leq (1 + K_8 \epsilon) |\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R},$$

where K_7 is a positive constant independent of ϵ , M and α , while K_8 is another positive constant independent of ϵ , M , α and β .

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We finally estimate the difference of w_i on two differential i -th characteristic at the same time.

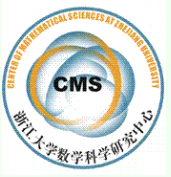
For arbitrary $\alpha, \beta \in \mathbb{R}$, we introduce

$$W_{\alpha, \beta}^*(\infty) = \max_{i=1, \dots, n} \sup_{t \in [0, \infty)} |w_i(t, x_i(t, \alpha)) - w_i(t, x_i(t, \beta))|. \quad (3.11)$$

Lemma 3.9 Under the assumptions of Theorem 1.1, there exists a positive constant K_9 independent of ϵ , M , α and β such that

$$W_{\alpha, \beta}^*(\infty) \leq (1 + K_9 \epsilon) \max_{i=1, \dots, n} |w_i(0, \alpha) - w_i(0, \beta)| + K_9 M^2 |\alpha - \beta|. \quad (3.12)$$

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Proof. By (2.9), it follows from (2.6) that

$$\begin{aligned}
 w_i(t, x_i(t, \alpha)) &= w_i(0, \alpha) + \sum_{j,k=1}^n \int_0^t (\gamma_{ijk}(u) w_j w_k)(s, x_i(s, \alpha)) ds \\
 &= w_i(0, \alpha) + \sum_{\substack{j \neq i \\ k \neq i}} \int_0^t [\gamma_{ijk}(u) w_j w_k](s, x_i(s, \alpha)) ds \\
 &\quad + \sum_{j \neq i} \int_0^t [\Delta_{ij}(u) w_i w_j](s, x_i(s, \alpha)) ds \\
 &\quad - \sum_{j \neq i} \int_0^t [\tilde{\Theta}_{ij}(u) w_i^2 u_j](s, x_i(s, \alpha)) ds,
 \end{aligned} \tag{3.13}$$

$$\tilde{\Theta}_{ij}(u) = - \int_0^1 \frac{\partial \gamma_{iii}(su_1, \dots, su_{i-1}, u_i, su_{i+1}, \dots, su_n)}{\partial u_j} ds.$$

$$\Delta_{ij}(u) = \gamma_{iji}(u) + \gamma_{iij}(u).$$

Remark $A(u) \in C^3$ is necessary to prove Lemma 3.9.

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4 | Asymptotic behavior of global classical solutions — Proof of Theorem 1.1

Let

$$\frac{D}{D_i t} = \frac{\partial}{\partial t} + \lambda_i(0) \frac{\partial}{\partial x}, \quad \alpha = x - \lambda_i(0)t. \quad (4.1)$$

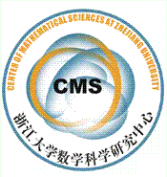
Noting (1.6a) and using Hadamard's formula, we have

$$\begin{aligned} \frac{D u_i}{D_i t} &= \sum_{j \neq i} (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i + (\lambda_i(0) - \lambda_i(u)) w_i r_i(u) e_i \\ &= \sum_{j \neq i} \{B_{ij}(u) w_j + \Gamma_{ij}(u) u_j w_i\}, \end{aligned} \quad (4.2)$$

Lemma 4.1 For $i \in \{1, \dots, n\}$ and any given $\alpha \in \mathbb{R}$, the limit

$$\lim_{t \rightarrow +\infty} u_i(t, \alpha + \lambda_i(0)t) = \Phi_i(\alpha) \quad (4.3)$$

exists; moreover, $|\Phi_i(\alpha)| \leq K_{10}\epsilon$.



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Lemma 4.2 For every $i \in \{1, \dots, n\}$, there exists a positive constant K_{12} independent of ϵ , M , α and β such that

$$|\Phi_i(\alpha) - \Phi_i(\beta)| \leq K_{12}M|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}. \quad (4.4)$$

Proof. For any $t \in \mathbb{R}^+$ and any $\alpha \in \mathbb{R}$ it holds that

$$u_i(t, \alpha + \lambda_i(0)t) = u_i(t, x_i(t, \theta_i(t, \alpha))). \quad (4.5)$$

Noting Lemma 3.8 and using (4.3), we have

$$\begin{aligned} \Phi_i(\alpha) - \Phi_i(\beta) &= \lim_{t \rightarrow \infty} u_i(t, \alpha + \lambda_i(0)t) - \lim_{t \rightarrow \infty} u_i(t, \beta + \lambda_i(0)t) \\ &= \lim_{t \rightarrow \infty} u_i(t, x_i(t, \theta_i(t, \alpha))) - \lim_{t \rightarrow \infty} u_i(t, x_i(t, \theta_i(t, \beta))) \\ &= \lim_{t \rightarrow \infty} u_i(t, x_i(t, \vartheta_i(\alpha))) - \lim_{t \rightarrow \infty} u_i(t, x_i(t, \vartheta_i(\beta))) \\ &= \lim_{t \rightarrow \infty} \{u_i(t, x_i(t, \vartheta_i(\alpha))) - u_i(t, x_i(t, \vartheta_i(\beta)))\}. \end{aligned} \quad (4.6)$$

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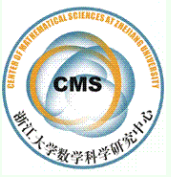
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Lemma 4.3 Suppose that $\lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t)$ exists, then

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t). \quad (4.7)$$

Proof.

$$\begin{aligned} \frac{d\Phi_i(\alpha)}{d\alpha} &= \lim_{t \rightarrow +\infty} \sum_{j=1}^n w_j(t, \alpha + \lambda_i(0)t) r_j(u(t, \alpha + \lambda_i(0)t)) e_i. \\ &= \lim_{t \rightarrow +\infty} \left\{ \sum_{j \neq i} w_j r_j(u) e_i + \sum_{j \neq i} O_{ij}(u) u_j w_i + w_i \right\} (t, \alpha + \lambda_i(0)t). \end{aligned} \quad (4.8)$$

$$u_j(t, \alpha + \lambda_i(0)t) \longrightarrow 0 \quad (j \neq i), \quad \text{as } t \rightarrow \infty$$

and

$$w_j(t, \alpha + \lambda_i(0)t) \longrightarrow 0 \quad (j \neq i), \quad \text{as } t \rightarrow \infty.$$

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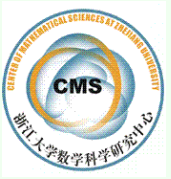
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Lemma 4.4 For any given $i \in \{1, \dots, n\}$ and for any fixed $\alpha \in \mathbb{R}$, the limit $\lim_{t \rightarrow +\infty} w_i(t, x_i(t, \alpha))$ exists, denoted it by $\Psi_i(\alpha)$, that is,

$$\lim_{t \rightarrow +\infty} w_i(t, x_i(t, \alpha)) = \Psi_i(\alpha), \quad \forall \alpha \in \mathbb{R}. \quad (4.9)$$

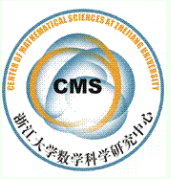
Moreover, $\Psi_i(\alpha)$ is a continuous function of $\alpha \in \mathbb{R}$ and satisfies that there exists a positive constant K_{13} independent of ϵ , M and α such that

$$|\Psi_i(\alpha)| \leq (1 + K_{13}\epsilon)M, \quad \forall \alpha \in \mathbb{R}. \quad (4.10)$$

In particular, if (1.13) is satisfied, then there exists a positive constant K_{14} independent of ϵ , M , ς , α and β such that

$$|\Psi_i(\alpha) - \Psi_i(\beta)| \leq K_{14}(\varsigma + M^2)|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}. \quad (4.11)$$

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Proof.

$$\begin{aligned}w_i(t, x_i(t, \alpha)) &= w_i(0, \alpha) + \sum_{\substack{j \neq i \\ k \neq i}} \int_0^t [\gamma_{ijk}(u) w_j w_k](s, x_i(s, \alpha)) ds + \\&\quad \sum_{j \neq i} \int_0^t [\Delta_{ij}(u) w_i w_j](s, x_i(s, \alpha)) ds - \\&\quad \sum_{j \neq i} \int_0^t [\tilde{\Theta}_{ij}(u) w_i^2 u_j](s, x_i(s, \alpha)) ds.\end{aligned}\tag{4.12}$$

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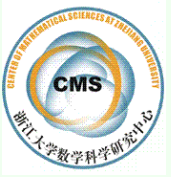
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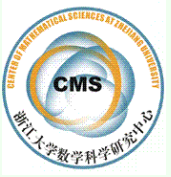
Lemma 4.5 For every $i \in \{1, \dots, n\}$, the limit $\lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t)$ exists, and

$$\lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t) = \Psi_i(\vartheta_i(\alpha)) \in C^0(\mathbb{R}). \quad (4.13)$$

Moreover, if (1.13) is satisfied, then the limit function $\Psi_i(\vartheta_i(\alpha))$ is globally Lipschitz continuous, and satisfies that, for every $\alpha, \beta \in \mathbb{R}$,

$$|\Psi_i(\vartheta_i(\alpha)) - \Psi_i(\vartheta_i(\beta))| \leq K_{14}(\varsigma + M^2)(1 + K_8\epsilon)|\alpha - \beta|. \quad (4.14)$$

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Proof. It follows from the definition of $\theta_i(t, \alpha)$ that

$$w_i(t, \alpha + \lambda_i(0)t) = w_i(t, x_i(t, \theta_i(t, \alpha))).$$

Then noting Lemma 3.8, we have

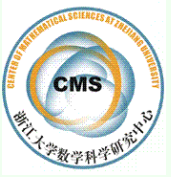
$$\lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t \rightarrow +\infty} w_i(t, x_i(t, \theta_i(t, \alpha))) = \lim_{t \rightarrow +\infty} w_i(t, x_i(t, \vartheta_i(\alpha))),$$

and then by (4.9),

$$\lim_{t \rightarrow +\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t \rightarrow +\infty} w_i(t, x_i(t, \vartheta_i(\alpha))) = \Psi_i(\vartheta_i(\alpha)) \quad (4.15)$$

Since $\Psi_i(\cdot)$ and $\vartheta_i(\cdot)$ are continuous with respect to $\cdot$ and $*$ respectively, $\Psi_i(\vartheta_i(\alpha))$ is a continuous function of $\alpha \in \mathbb{R}$. This proves (4.13). $\square$

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Combining Lemmas 4.3 and 4.5 gives

Lemma 4.6 For every $i \in \{1, \dots, n\}$, it holds that

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \Psi_i(\vartheta_i(\alpha)) \in C^0(\mathbb{R}). \quad (4.16)$$

Moreover, if (1.13) is satisfied, then the following estimate hold

$$\left| \frac{d\Phi_i}{d\alpha}(\alpha) - \frac{d\Phi_i}{d\alpha}(\beta) \right| \leq K_{15}(\varsigma + M^2)|\alpha - \beta|, \quad \forall \alpha, \beta \in \mathbb{R}, \quad (4.17)$$

where K_{15} is a positive constant independent of ϵ , M , ς , α and β .

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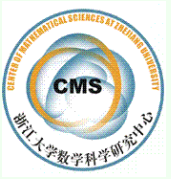
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Proof of Theorem 1.1 Taking

$$\varphi_i(x - \lambda_i(0)t) = \Phi_i(x - \lambda_i(0)t) \quad (i = 1, \dots, n), \quad (4.18)$$

and noting Lemmas 4.1-4.2 and Lemma 4.6, we get the conclusion of Theorem 1.1 immediately. Thus, the proof of Theorem 1.1 is completed. $\square$

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